

Optimal and covariant single-copy LOCC transformation between two two-qubit states

K. Brádler

*Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, México 01000, México**

(Dated: February 1, 2008)

Given two two-qubit pure states characterized by their Schmidt numbers we investigate an optimal strategy to convert the states between themselves with respect to their local unitary invariance. We discuss the efficiency of this transformation and its connection to LOCC convertibility properties between two single-copy quantum states. As an illustration of the investigated transformations we present a communication protocol where in spite of all expectations a shared maximally entangled pair between two participants is the worst quantum resource.

PACS numbers: 03.67.Mn

Keywords: LOCC transformation, covariant quantum channels, semidefinite programming

I. INTRODUCTION

One of the greatest achievements of quantum information theory (QIT) is the realization that quantum entanglement serves as a resource for performing various communication tasks where Ekert's scheme [1] for quantum key distribution (QKD) or quantum teleportation [2] are the most flagrant examples. Shortly after, the question of equivalence of different multipartite states came into question. Partially motivated by the security issues in QKD (i.e. how to locally distill a shared non maximally entangled or even noised quantum state to avoid any correlations with a potential eavesdropper) the problem of LOCC [4] (local operation and classical communication) convertibility [5] became fundamental. We can approach the question from two extremal sides. Namely, asking whether two states are LOCC transformable in an asymptotic limit or having just a single copy of an initial state at our disposal. Both approaches brought the considerable progress in QIT. To name just few, in the first regime, several measures of entanglement were defined in terms of an asymptotic rate in which it is possible to convert from/to a maximally entangled state [3, 4]. In the second case, the connection between the Schmidt number majorization [6] and LOCC state transformation was discovered [17] or new classes of tripartite entangled states were presented [7].

We will treat with an interesting QIT paradigm which is so called impossibility transformation (or 'no-go process'). There exist several kinds of impossible transformations stratified by the fact how the impossibility is fundamental. Quantum cloning [9] or finding the orthogonal complement to a given quantum state (universal NOT) [10] belong to the group of the highest stratum. This kind of impossibility comes from basic principles of quantum mechanics [8] and can be performed just approximately [9, 10]. There are also known other examples of fundamentally impossible processes [11]. In the lower level there exist transformations which are not forbidden by the laws of quantum mechanics but they are impossible under some artificially augmented requirements. Typically, we consider only LOCC operations as, for example, the above mentioned single-copy transformation task [17]. In this case, without the LOCC constraint there is no problem to transform one pure state to another without any limitations.

In this paper we use the methods of semidefinite programming [20] to find an optimal and completely positive (CP) map for LOCC single-copy pure state transformation regarding its covariant properties. Covariance means that the sought CP maps are universal in the sense that they do not change their forms under the action of $SU(2)$ group (or their products) on the input states. The covariance requirement was also added to other quantum mechanical processes, compare e.g. [22]. In addition to the covariance, we require optimality meaning that the output state produced by the CP LOCC map is maximally close to the required target state. The closeness is measured by the value of the fidelity between the actual output state and the desired target state. As we will see, our problem of covariant and optimal LOCC state transformation combines both kinds of the impossibilities mentioned above.

The structure of the paper is the following. In section II we recall some basics facts about the isomorphism between quantum maps and the related group properties. The main part of this paper can be found in section III where the optimal LOCC single-copy state transformation is investigated with the help of semidefinite programming techniques. Section IV can be regarded as an application of the studied problem where we present a communication protocol for the LOCC transmission of a local unitary operation from one branch of a shared two-qubit state to the second one. We show that a maximally entangled pair does not always need to be the best quantum communication resource. The corresponding Kraus maps for the protocol are listed in Appendix.

*Electronic address: kbradler@epot.cz

II. METHODS

It is well known that there exists an isomorphism between completely positive maps $\mathcal{M}(\varrho)$ and semidefinite operators $R_{\mathcal{M}}$, first introduced by Jamiołkowski [12]

$$\mathcal{M}(\varrho_{in}) = \text{Tr}_{in} [(\mathbb{1} \otimes \varrho_{in}^T) R_{\mathcal{M}}] \iff R_{\mathcal{M}} = (\mathcal{M} \otimes \mathbb{1})(P^+), \quad (1)$$

where $P^+ = \sum_{i,j=0}^{d-1} |ii\rangle\langle jj|$ is a maximally entangled bipartite state of the dimension d^2 . The isomorphism allows us to fulfill otherwise a difficult task of the parametrization of all CP maps by putting a positivity condition on the operator $R_{\mathcal{M}}$. Then, the parametrization problem is computationally much more feasible. This is not the only advantage the representation offers. As was shown in [15], the representation is useful for the description of quantum channels (so called covariant channels) which we wish to optimize regarding some symmetry properties. More precisely, having two representations V_1, V_2 of a unitary group, the map $\mathcal{M}(\varrho)$ is said to be covariant if $\mathcal{M}(\varrho) = V_2^\dagger \mathcal{M}(V_1 \varrho V_1^\dagger) V_2$. Inserting the covariance condition into Eq. (1) and using the fact that the positive operator $R_{\mathcal{M}}$ is unique, we get

$$R_{\mathcal{M}} = (V_2^\dagger \otimes V_1^T) R_{\mathcal{M}} (V_2 \otimes V_1^*) \iff [R_{\mathcal{M}}, V_2 \otimes V_1^*] = 0. \quad (2)$$

The space occupied by $V_2 \otimes V_1^*$ can be decomposed into a direct sum of irreducible subspaces and from Schur's lemma follows that $R_{\mathcal{M}}$ is a sum of the isomorphisms between all equivalent irreducible representations. If we now consider the fidelity equation in the Jamiołkowski representation

$$F = \text{Tr} [(\varrho_{out} \otimes \varrho_{in}^T) R_{\mathcal{M}}] \quad (3)$$

the task is reduced on finding the maximum of F subject to non-negativity of $R_{\mathcal{M}}$ and other constraints posed on $R_{\mathcal{M}}$. In our case, it is the trace preserving condition $\text{Tr}_{in} [R_{\mathcal{M}}] = \mathbb{1}$ following from (1). This can be easily reformulated as a semidefinite program [20] and thus efficiently solved using computers. Moreover, it is easy to put other conditions on $R_{\mathcal{M}}$ such as partial positive transpose condition (PPT) and they can be easily implemented as well [21]. Recall that for two-qubit systems the PPT condition is equivalent to the LOCC requirement. Note that the usefulness of the presented method was already shown, for example, in connection with optimal and covariant cloning [16].

In our calculation we employed the YALMIP environment [24] equipped with the SeDuMi solver [25]. One of the advantages of semidefinite programming is the indication of which parameters are zero. Then, analytical solutions for the fidelity and even general forms of the Kraus decomposition [26] of the CP map may be found. In our problem, using the properties of the Jamiołkowski positive matrix $R_{\mathcal{M}}$ (which are stated as an almost computer-ready theorem in [13, 14]) we derived the corresponding Kraus operators as general as possible.

III. OPTIMAL AND COVARIANT SINGLE-COPY LOCC STATE TRANSFORMATION

Let us have an input and target state written in their Schmidt forms $|\chi\rangle = a|00\rangle + \sqrt{1-a^2}|11\rangle$, $|\varphi\rangle = c|00\rangle + \sqrt{1-c^2}|11\rangle$; $a, c \in (0, 1/\sqrt{2})$. It was shown [17] that a deterministic LOCC conversion $|\chi\rangle \rightarrow |\varphi\rangle$ is possible iff $a \geq c$. If we want to go in the direction where LOCC is not powerful enough we have basically two strategies at our disposal. First, in some cases we may choose a probabilistic strategy [18] also called conclusive conversion. As an alternative, there exists a possible LOCC deterministic transformation to a state which is in some sense closest to the required one [19]. Namely, it is a state in which the fidelity with the target state is maximal. We will follow a related way and find how an optimal covariant LOCC CP map best approximates the ideal transformation $|\chi\rangle \leftrightarrow |\varphi\rangle$. In the next sections, we consider the following parameter space $a, c \in (0, 1)$ both for $|\chi\rangle$ and $|\varphi\rangle$.

Adopting the covariance considerations from the previous section into our case we demand

$$F = \langle \varphi | \mathcal{M}(|\chi\rangle\langle\chi|) | \varphi \rangle = \langle \varphi' | \mathcal{M}(|\chi'\rangle\langle\chi'|) | \varphi' \rangle = F', \quad (4)$$

where $V_1 |\chi\rangle = |\chi'\rangle$, $V_2 |\varphi\rangle = |\varphi'\rangle$ and the covariance condition (2) follows (note that quite accidentally the condition is the same as in case of covariant cloning).

A. LOCC semicovariant transformation

Firstly, we will be interested in how $|\chi\rangle$ can be transformed if $V_1 = V_2 = \mathbb{1} \otimes U$ where U is a unitary representation of $SU(2)$. In other words, we consider the situation where the covariance is imposed on one branch of $|\chi\rangle$ (we call it a *semicovariant* case). From Eq. (2) follows

$$[R_{\mathcal{M}}, \mathbb{1} \otimes U \otimes \mathbb{1} \otimes U^*] = 0 \iff [\tilde{R}_{\mathcal{M}}, \mathbb{1} \otimes \mathbb{1} \otimes U \otimes U] = 0, \quad (5)$$

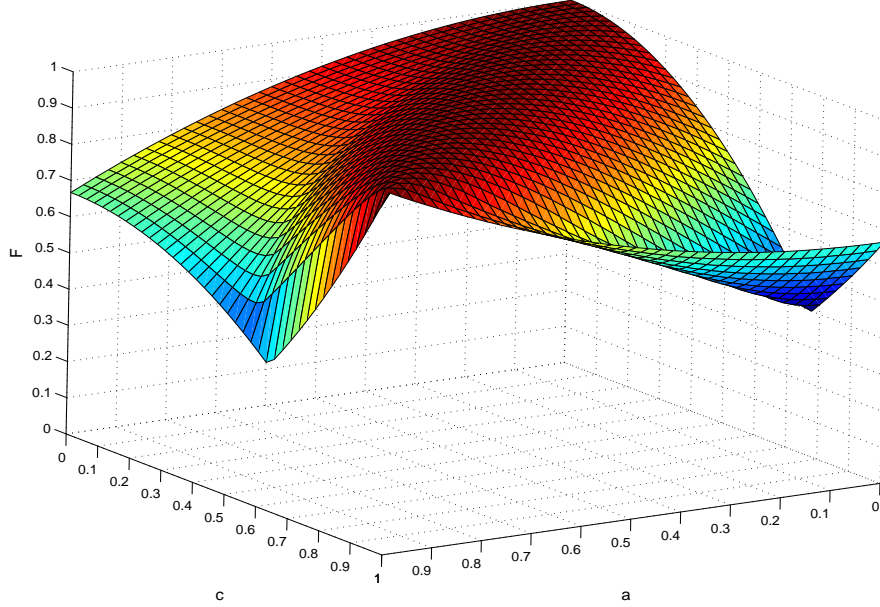


FIG. 1: The fidelity for the optimal and locally semicovariant LOCC transformation between $|\chi\rangle = a|00\rangle + \sqrt{1-a^2}|11\rangle$ and $|\varphi\rangle = c|00\rangle + \sqrt{1-c^2}|11\rangle$.

where $R_{\mathcal{M}} = S^\dagger \tilde{R}_{\mathcal{M}} S$ and $S = \mathbb{1} \otimes \text{SWAP} \otimes \sigma_Y$ where $\text{SWAP} = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$ and σ_Y is the Pauli Y operator. With the unitarily transformed rhs in Eq. (5) the decomposition is found in a particularly simple way

$$\tilde{R}_{\mathcal{M}} = \bigoplus_{i,j=1}^4 s_{ij} P_{S_{ij}} \oplus a_{ij} P_{A_{ij}}, \quad (6)$$

where $P_{S_{ij}}, P_{A_{ij}}$ are isomorphisms between equivalent symmetrical and antisymmetrical irreducible subspaces, respectively. There are 32 free complex parameters but we know that $\tilde{R}_{\mathcal{M}}$ is a nonnegative operator. It follows that a_{ii}, s_{ii} are real and $a_{ij} = a_{ji}^*, s_{ij} = s_{ji}^*$. The number of free parameters is thus reduced to 32 real numbers. Maximizing the fidelity (3) for $\varrho_{in} = |\chi\rangle\langle\chi|, \varrho_{out} = |\varphi\rangle\langle\varphi|$ with this number of parameters is far from a possible analytical solution but feasible in terms of semidefinite programming. For $i \neq j$ it is advantageous to introduce the decomposition $a_{ij} P_{A_{ij}} + a_{ji}^* P_{A_{ji}} = \Re[a_{ij}](P_{A_{ij}} + P_{A_{ji}}) + \Im[a_{ij}](iP_{A_{ij}} - iP_{A_{ji}})$ and similarly for the symmetrical part. With the above defined variables the fidelity to be maximized has the form (leaving out the zero parameters)

$$F = \frac{1}{2} \left(a^2 c^2 (s_{11} + a_{11}) + (1 - c^2)(1 - a^2)(s_{44} + a_{44}) + c^2(1 - a^2)s_{22} + (1 - c^2)a^2 s_{33} + ac\sqrt{(1 - a^2)(1 - c^2)}a_7^+ \right), \quad (7)$$

where $a_7^+ = \Re[a_{41}]$. The resulting fidelity is depicted in Fig. 1. First, we note that for $a \leq c$ the result corresponds to the analytical result found in [19] which, for our bipartite case, has the form

$$F = \left(ac + \sqrt{(1 - a^2)(1 - c^2)} \right)^2. \quad (8)$$

The reason is that the optimal fidelity found in [19] is dependent only on the Schmidt numbers of the input and target state and thus it is automatically locally covariant. If we do not consider the parameters of $R_{\mathcal{M}}$ which are shown to be zero (yielded from the semidefinite program) a general form in the Kraus representation can be in principle found ($R_{\mathcal{M}}$ can be diagonalized with the help of a software for the symbolic manipulations). But it appears that this decomposition is too complex and for our purpose it is not necessary to present it. The only comment is deserved by the identity map which covers the whole region of parameters where Eq. (8) is valid. This is in contrast with the original work [19] where the map is not the identity due to the knowledge of parameters a, c . In reality, this trivial map appears to be the covariant and optimal map for a bit larger region as depicted in Fig. 2. It follows that under the realm of the identity map no optimal covariant CP map exists. The remaining part of the parameter space

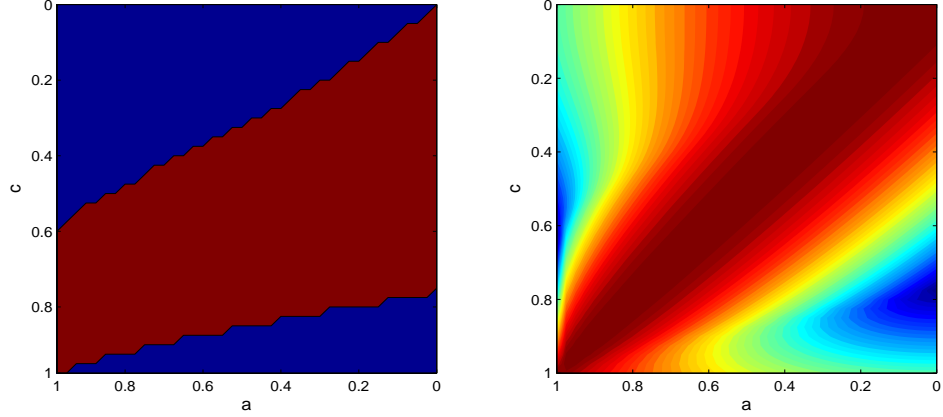


FIG. 2: A 2D view on Fig. 1 (on the right) together with the indication where the trivial identity is the optimal map (the red area on the left). The blue part corresponds to various non-unit maps.

of a, c shows that in spite of the allowance of the perfect deterministic conversion by the majorization criterion the semicovariant transformation does not reach the maximal fidelity. We intentionally left out the word LOCC because the second interesting aspect is that for the whole parameter space the LOCC condition is unnecessary. In other words, there are only LOCC semicovariant transformations or the identity map which is also (trivially) LOCC semicovariant. We confirm the existence of another fundamental no-go process saying that it is not possible to construct a CP map perfectly copying a partially or totally unknown quantum state to a generally different quantum state even if the majorization criterion allows us to do it (attention to the related problem was called in [23]). The impossibility is easy to show by considering the following tiny lemma valid not only for the investigated dimension $d = 2$:

Let M be a unitary and covariant map, i.e. $|\varphi\rangle = M|\chi\rangle$ holds for two arbitrary qudits $|\chi\rangle, |\varphi\rangle$. Then, from the covariance follows $MU|\chi\rangle = U|\varphi\rangle = UM|\chi\rangle \iff [M, U] = 0$. We suppose that this holds for all $U \in SU(d)$ and then by one of Schur's lemma $M = c\mathbb{1}$. Considering the requirement of unitarity of M it follows $c = 1$ and thus $|\varphi\rangle = |\chi\rangle$. \square

We confirmed this lemma in Fig. 1 where the fidelity is equal to one only if $a = c$ and we may reflect the calculated optimal values of the fidelity as a refinement and quantification how much is the above process impossible.

Note that the majorization criterion [17] was developed with respect to the degree of entanglement (the Schmidt number) but relies on the complete knowledge of the converted state what is at variance with the covariant requirement where no particular state is preferred. The situation is a bit similar to quantum cloning where if we know the preparation procedure of a state to be cloned then there is no problem to make an arbitrary number of its perfect copies.

Another worthy aspect is that the interval of a and c goes from zero to one thus covering the target states with the same Schmidt number more than once. Nevertheless, the fidelity is different in such cases (compare e.g. the target states $|00\rangle$ and $|11\rangle$). In fact, to completely describe the (semi)covariant properties of the type presented in this article we should not distinguish input and target states by their Schmidt numbers but rather to fully parametrize them in $SU(2) \otimes SU(2)$ representation for every $a, c \in (0, 1/\sqrt{2})$. But by relying on the lemma above we expect that this situation does not bring anything surprising into our discussion. Also, due to the (semi)covariance we have actually described potentially interesting transformations between $|\chi\rangle = a|01\rangle + \sqrt{1-a^2}|10\rangle$ and $|\varphi\rangle = c|01\rangle + \sqrt{1-c^2}|10\rangle$.

B. Full LOCC covariant transformations

As the second case we investigate a full local covariance where, first, both qubits from an input two-qubit state $|\chi\rangle$ are rotated simultaneously and, second, both qubits are rotated independently. The covariance with respect to these two types of rotation is required.

The covariance condition in the first case is $V_1 = V_2 = U \otimes U$ and thus

$$[R_{\mathcal{M}}, U \otimes U \otimes U^* \otimes U^*] = 0 \iff [\tilde{R}_{\mathcal{M}}, U \otimes U \otimes U \otimes U] = 0. \quad (9)$$

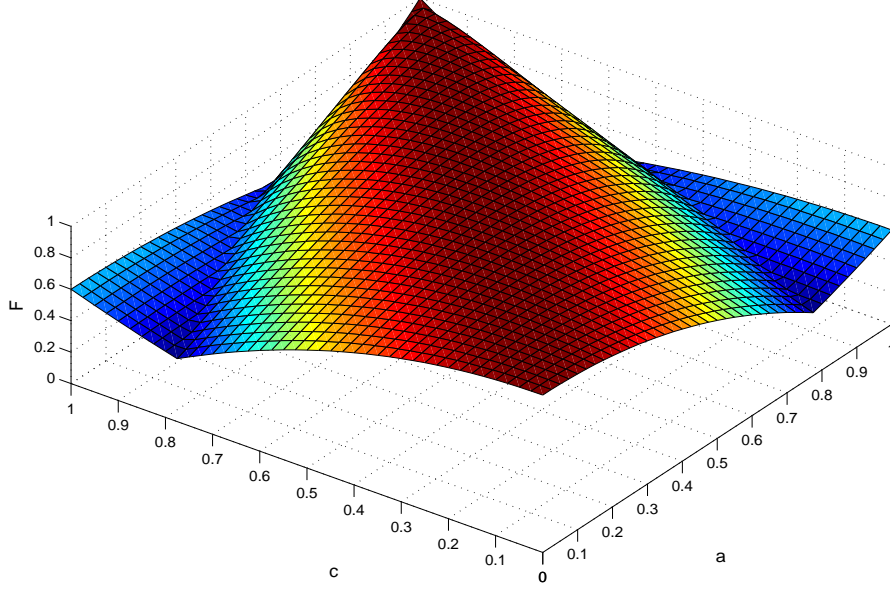


FIG. 3: The fidelity for the optimal and full locally covariant LOCC transformation between $|\chi\rangle = a|00\rangle + \sqrt{1-a^2}|11\rangle$ and $|\varphi\rangle = c|00\rangle + \sqrt{1-c^2}|11\rangle$.

Employing the fact that

$$SU(2)_{j=1/2}^{\otimes 4} = \bigoplus_{J=0}^2 c_J D^{(J)} \quad (10)$$

with $c_J \in (2, 3, 1)$ we find the basis vectors of all irreducible subspaces (summarized in Tab. I) and construct isomorphisms P between equivalent species

$$\tilde{R}_{\mathcal{M}} = \bigoplus_{J=0}^2 \bigoplus_{k,l=1}^{c_J} d_{Jkl} P_{D_{kl}^{(J)}}. \quad (11)$$

Choosing the parameters d_{Jkl} we require $R_{\mathcal{M}}$ to be a semidefinite matrix. We calculate the fidelity for the same kind of input/target states from the previous subsection yielding

$$F = \left(ac + \sqrt{(1-a^2)(1-c^2)} \right)^2 \left(\frac{1}{3} d_{022} + \frac{1}{6} d_{211} \right) + (c^2(1-a^2) + (1-c^2)a^2) d_{211}. \quad (12)$$

Running an appropriate semidefinite program for maximizing F we are able to get analytical results both for the fidelity and the CP map in the Kraus form. It appears that many of the coefficients d_{Jkl} are zero and thus Eq. (12) simplifies as well as the constraints given by the trace preserving condition. As far as the LOCC condition the situation here is that the CP maps with and without the posed condition are different but both give the same optimal fidelity. It can be shown that the LOCC condition in this case is just a dummy constraint determining the value of a free parameter in the resulting map (see the parameter d_{011} in Eq. (14)). Then

$$F = \max \left[\left(ac + \sqrt{(1-a^2)(1-c^2)} \right)^2, \frac{1}{10} \left(ac + \sqrt{(1-a^2)(1-c^2)} \right)^2 + \frac{3}{5} (c^2(1-a^2) + a^2(1-c^2)) \right] \quad (13)$$

and the corresponding graph is in Fig. 3. It is noteworthy that there are just two types of CP covariant maps for two investigated intervals of a, c corresponding to the different fidelity functions in (13). The identity map is the first one

TABLE I: Orthogonal basis vectors of all irreducible subspaces of $SU(2)_{j=1/2}^{\otimes 4}$.

Total momentum J	Irreducible subspace $D_{kl}^{(J)}$	Basis vectors
0	$D_{11}^{(0)}$	$\frac{1}{2} 01 - 10\rangle 01 - 10\rangle$
0	$D_{22}^{(0)}$	$\frac{1}{\sqrt{3}} (0011\rangle - \frac{1}{2} 01 + 10\rangle 01 + 10\rangle + 1100\rangle)$
1	$D_{11}^{(1)}$	$\frac{1}{\sqrt{2}} 01 - 10\rangle 00\rangle$ $\frac{1}{2} 01 - 10\rangle 01 + 10\rangle$ $\frac{1}{\sqrt{2}} 01 - 10\rangle 11\rangle$
1	$D_{22}^{(1)}$	$\frac{1}{\sqrt{2}} 00\rangle 01 - 10\rangle$ $\frac{1}{2} 01 + 10\rangle 01 - 10\rangle$ $\frac{1}{\sqrt{2}} 11\rangle 01 - 10\rangle$
1	$D_{33}^{(1)}$	$-\frac{1}{2} (00\rangle 01 + 10\rangle - 01 + 10\rangle 00\rangle)$ $-\frac{1}{\sqrt{2}} (0011\rangle - 1100\rangle)$ $-\frac{1}{2} (01 + 10\rangle 11\rangle - 11\rangle 01 + 10\rangle)$
2	$D_{11}^{(2)}$	$ 0000\rangle$ $\frac{1}{2} (00\rangle 01 + 10\rangle + 01 + 10\rangle 00\rangle)$ $\frac{1}{\sqrt{6}} (0011\rangle + 1100\rangle + 01 + 10\rangle 01 + 10\rangle)$ $\frac{1}{2} (01 + 10\rangle 11\rangle + 11\rangle 01 + 10\rangle)$ $ 1111\rangle$

and the conclusion from the previous case holds. The second map is described by the set of the Kraus operators

$$\begin{aligned}
A_1 &= \sqrt{\frac{1-d_{011}}{3}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \sqrt{\frac{1-d_{011}}{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, A_3 = \sqrt{\frac{1-d_{011}}{12}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
A_4 &= \frac{\sqrt{d_{011}}}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_5 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_6 = \sqrt{\frac{3}{20}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\
A_7 &= \sqrt{\frac{3}{20}} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_8 = \sqrt{\frac{3}{5}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_9 = \sqrt{\frac{3}{5}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{14}
\end{aligned}$$

where d_{011} is a free parameter from the decomposition (11). The trace-preserving condition $\sum_{i=1}^9 A_i^\dagger A_i = \mathbb{1}$ is satisfied [28].

Let us proceed to the second case where we consider independent unitary rotations on both qubits of the pair, that is $V_1 = V_2 = U_1 \otimes U_2$. Derived analogously as before, it follows

$$[\tilde{R}_{\mathcal{M}}, U_1 \otimes U_1 \otimes U_2 \otimes U_2] = 0 \tag{15}$$

with the decomposition in a particularly simple form

$$\tilde{R}_{\mathcal{M}} = p_1 P_A \otimes P_A + p_2 P_A \otimes P_S + p_3 P_S \otimes P_A + p_4 P_S \otimes P_S, \tag{16}$$

where P_A, P_S are the projectors into asymmetrical and symmetrical subspaces, respectively [27]. The resulting fidelity equation (again independent on the LOCC condition) can be derived analytically

$$F = \max \left[\left(ac + \sqrt{(1-a^2)(1-c^2)} \right)^2, \frac{1}{9} \left(ac + \sqrt{(1-a^2)(1-c^2)} \right)^2 + \frac{4}{9} (c^2(1-a^2) + a^2(1-c^2)) \right] \tag{17}$$

with the picture looking similarly as in Fig. 3. The achieved fidelity is even lower due to the stronger requirements on the covariance properties in Eq. (15) in comparison with Eq. (9). As in the previous case, there are two maps for two

different fidelity functions, one of them being the identity map. The Kraus decomposition of the nontrivial map is

$$A_1 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{3} & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{3} & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (18)$$

and $A_6 = A_2^\dagger, A_7 = A_3^\dagger, A_8 = A_5^\dagger, A_9 = A_4^\dagger$.

IV. COVARIANT LOCC COMMUNICATION PROTOCOL

Let us try to apply the previous considerations to the solution of the following communication problem. Suppose that the two-qubit state $|\chi\rangle = a|00\rangle + \sqrt{1-a^2}|11\rangle$ was locally and unitarily modified on Alice's side and then distributed between Alice and Bob. Next imagine that the distributor of this state is confused and oblivious and he wanted originally to modify Bob's part of the state. Moreover, he forgot which unitary modification was done. Since Alice and Bob are separated the only possibility to rectify the distributor's mistake is LOCC communication between them. In other words, they would like to perform the following transformation

$$|\chi'\rangle = (U \otimes \mathbb{1})|\chi\rangle \xrightarrow{LOCC} (\mathbb{1} \otimes U)|\chi\rangle = |\varphi'\rangle \quad (19)$$

such that the LOCC transformation will be equally and maximally successful irrespective of U . Generally, this is the problem of sending an unknown local unitary operation between branches of a shared bipartite state. Notice that if $|\chi\rangle$ is a maximally entangled state then the task changes to finding a transposition of the unitary operation U due to the well known relation

$$(U \otimes \mathbb{1})|00 + 11\rangle = (\mathbb{1} \otimes U^T)|00 + 11\rangle. \quad (20)$$

The covariant condition in the Jamiolkowski representation reads

$$[R_{\mathcal{M}}, \mathbb{1} \otimes U \otimes U^* \otimes \mathbb{1}] = 0 \quad (21)$$

using decomposition (6) and the unitary modification $\tilde{R}_{\mathcal{M}} = SR_{\mathcal{M}}S^\dagger$ with $S = (\mathbb{1} \otimes SWAP \otimes \sigma_Y)(\mathbb{1} \otimes \mathbb{1} \otimes SWAP)$. Again, the figure of merit is the fidelity which now has the form

$$F = \frac{1}{2} (a^4(s_{11} + a_{11}) + (1 - a^2)^2(s_{44} + a_{44})) + a^2(1 - a^2)(s_{22} + s_{33} + a_7^+ - s_7^+), \quad (22)$$

where $a_7^+ = \Re[a_{41}]$, $s_7^+ = \Re[s_{41}]$. One may find a general form of this map in terms of the Kraus operators in Appendix. If we first run the corresponding semidefinite program without the LOCC condition we get the fidelity equal to one for all a . This has a reasonable explanation because if we allow the nonlocal operations there exists a universal and always successful unitary operation – SWAP. The inspection of the particular $R_{\mathcal{M}}$ confirms this inference. After imposing the LOCC condition the resulting fidelity is depicted in Fig 4. This result is noteworthy because we see that the LOCC CP map is the most successful for the factorized states ($a = 0, 1 \sim F = 2/3$) while it holds $F = 1/2$ for the maximally entangled states. The reason lies in Eqs. (20) and (19). If $|\chi\rangle$ is a maximally entangled state then a local unitary action passes the whole local orbit whereas for non-maximally entangled states the unitary action on one branch is not sufficient for the attainment of all possible partially entangled states characterized by the same Schmidt number a . We may conclude with an intriguing claim that in case of our protocol it is better for Alice and Bob to share a factorized state instead of a maximally entangled state. Let us stress that the optimal map is not trivially identical for any value of the parameter a in the input state $|\chi\rangle$.

V. CONCLUSION

In this work we studied the LOCC transformations between two-qubit bipartite states characterized by their Schmidt numbers. In addition to the obvious CP requirement, we looked for the covariant maps which maximize the fidelity between an input and a target state. Moreover, we supposed that we had just a single copy of the input state at our disposal. The studied covariance can be divided into two groups: so called semicovariance where we required the independence of the input state regarding the action of $SU(2)$ representation on one of the input qubits. The second investigated possibility were two cases of full covariance condition where the independence and optimality of the state

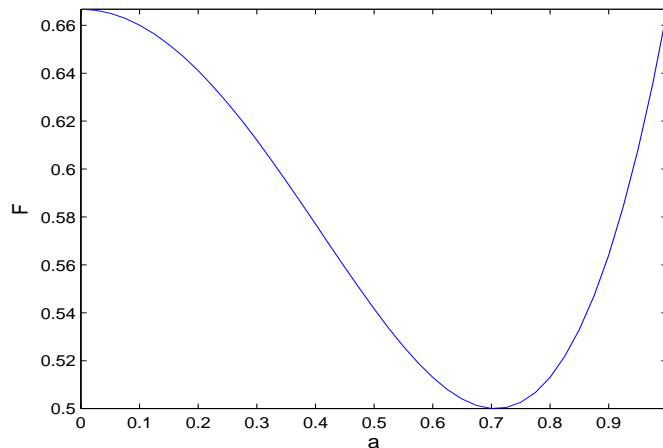


FIG. 4: The fidelity of the protocol for ‘handing over’ a local unitary operation between branches of a partially entangled two-qubit pair. The entanglement of the shared pair is characterized by the Schmidt number a .

transformation had been examined with respect to two (equivalent and nonequivalent) $SU(2)$ representations acting on both branches of the input bipartite state.

We employed the methods of semidefinite programming which, in spite of being a numerical method, enables us to find totally or partially general analytical solutions for the fidelity and for the corresponding LOCC CP maps. We have found that, first, due to the covariance conditions there are no possible perfect state transformations even if the majorization criterion allows them and with the calculated optimal fidelity we quantified the ‘maximal allowance’ of the considered transformations. Second, we have shown that there only exist LOCC covariant transformations. Hence, since this condition is unnecessary this kind of transformation can be rated as another basic process forbidden by the laws of quantum mechanics. We have also connected our work with the earlier works on so called faithful single-copy state transformations [19]. Notably, for the corresponding subset of the investigated parameter area the same analytical results for the fidelity were derived but under the local unitary covariant circumstances. Consequently, the forms of the particular CP maps are different from previously derived putting this problem into a different perspective.

Finally, we illustrated these methods on an application of the communication protocol for LOCC ‘handing over’ of a local unitary operation from one branch of a shared two-qubit bipartite state to another without its actual knowledge. Intriguingly, it has been shown that the best results (in terms of the fidelity between an input and a target state) are achieved if both parties share one of the considered factorized states $|00\rangle$ or $|11\rangle$ and not the maximally entangled state.

Even if for general multipartite states the PPT condition used here is not equivalent to the LOCC condition, the described methods might be useful for this kind of study as well, for example, to help clarifying the role of the PPT operations and the transformation properties of these states.

Acknowledgments

The author is very indebted for discussions and support from R. Jáuregui and for comments from R. Demkowicz-Dobrzański.

APPENDIX A

Considering

$$p_{1,2} = \frac{-s_{11} + s_{44} \pm \sqrt{s_{11}^2 - 2s_{11}s_{44} + s_{44}^2 + 4(s_7^+)^2}}{2s_7^+} \quad (\text{A1})$$

$$p_{3,4} = \frac{-a_{11} + a_{44} \pm \sqrt{a_{11}^2 - 2a_{11}a_{44} + a_{44}^2 + 4(a_7^+)^2}}{2a_7^+} \quad (\text{A2})$$

and

$$d_1 = \frac{1}{\sqrt{2}} \left(s_{11} + s_{44} + \sqrt{s_{11}^2 - 2s_{11}s_{44} + s_{44}^2 + 4(s_7^+)^2} \right)^{1/2} \quad (\text{A3})$$

$$d_2 = \frac{1}{\sqrt{2}} \left(s_{11} + s_{44} - \sqrt{s_{11}^2 - 2s_{11}s_{44} + s_{44}^2 + 4(s_7^+)^2} \right)^{1/2} \quad (\text{A4})$$

$$d_3 = \frac{1}{\sqrt{2}} \left(a_{11} + a_{44} + \sqrt{a_{11}^2 - 2a_{11}a_{44} + a_{44}^2 + 4(a_7^+)^2} \right)^{1/2} \quad (\text{A5})$$

$$d_4 = \frac{1}{\sqrt{2}} \left(a_{11} + a_{44} - \sqrt{a_{11}^2 - 2a_{11}a_{44} + a_{44}^2 + 4(a_7^+)^2} \right)^{1/2} \quad (\text{A6})$$

$$d_5 = \sqrt{s_{22}} \quad (\text{A7})$$

$$d_6 = \sqrt{s_{33}} \quad (\text{A8})$$

we may write the Kraus operators for the problem in Sec. IV as

$$\begin{aligned} A_1 &= \frac{d_1}{\sqrt{2}} \frac{1}{\sqrt{1+p_1^2}} \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & 0 & -p_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \frac{d_1}{\sqrt{2}} \frac{1}{\sqrt{1+p_1^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ A_4 &= \frac{d_2}{\sqrt{2}} \frac{\text{sign}(p_2-p_1)}{\sqrt{1+p_1^2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & 0 & -p_1 \end{pmatrix}, A_5 = \frac{d_2}{\sqrt{2}} \frac{\text{sign}(p_2-p_1)}{\sqrt{1+p_1^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ A_7 &= \frac{d_3}{\sqrt{2}} \frac{1}{\sqrt{1+p_3^2}} \begin{pmatrix} p_3 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_8 = \frac{d_3}{\sqrt{2}} \frac{\text{sign}(p_3-p_4)}{\sqrt{1+p_3^2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}, \\ A_9 &= \frac{d_5}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_{10} = d_5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_{11} = d_5 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_{12} &= \frac{d_6}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_{13} = d_6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_{14} = d_6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A9})$$

and $A_3 = A_2^\dagger, A_6 = A_5^\dagger$. The maps satisfy $\sum_{i=1}^{14} A_i^\dagger A_i = \mathbb{1}$ if the trace preserving condition on the Jamiołkowski map is posed. Similarly to Eq. (14), the Kraus operators are not in their apparent LOCC form but can be transformed into it.

-
- [1] A. K. Ekert Phys. Rev. Lett. **67**, 661 (1991)
 - [2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters Phys. Rev. Lett. **70**, 1895 (1993)
 - [3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters Phys. Rev. A **54**, 3824 (1996)
 - [4] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher Phys. Rev. A **53**, 2046 (1996) C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters Phys. Rev. Lett. **76**, 722 (1996)
 - [5] L. Hardy Phys. Rev. A **60**, 1912 (1999) D. Jonathan and M. B. Plenio Phys. Rev. Lett. **83**, 3566 (1999) D. Jonathan and M. B. Plenio Phys. Rev. Lett. **83**, 1455 (1999)
 - [6] R. Bhatia, Matrix Analysis (Springer-Verlag, New York, 1997)
 - [7] W. Dür, G. Vidal, and J. I. Cirac Phys. Rev. A **62**, 062314 (2000)
 - [8] W. K. Wootters and W. H. Zurek Nature **299**, 802 (1982)

- [9] V. Bužek and M. Hillery Phys. Rev. A **54**, 1844 (1996)
- [10] V. Bužek, M. Hillery and R. F. Werner Phys. Rev. A **60**, R2626 (1999) N. Gisin and S. Popescu Phys. Rev. Lett. **83**, 432 (1999)
- [11] A. K. Pati and S. L. Braunstein Phys. Lett. A **315**, 208 (2003)
- [12] A. Jamiolkowski Rep. Math. Phys. **3**, 275 (1972)
- [13] D. Salgado, J. Sánchez-Gómez, M. Ferrero Open Sys. & Information Dyn. **12**, 55 (2005)
- [14] I. Bengtsson and K. Życzkowski, Geometry of Quantum States. An Introduction to Quantum Entanglement (Cambridge University Press, 2006)
- [15] G. M. D'Ariano and P. Lo Presti Phys. Rev. A **64**, 042308 (2001)
- [16] R. Demkowicz-Dobrzański, M. Kuś, and K. Wódkiewicz Phys. Rev. A **69**, 012301 (2004), R. Demkowicz-Dobrzański, M. Lewenstein, A. Sen(De), U. Sen, and D. Bruß Phys. Rev. A **73**, 032313 (2006)
- [17] M. A. Nielsen Phys. Rev. Lett. **83**, 436 (1999)
- [18] G. Vidal Phys. Rev. Lett. **83**, 1046 (1999)
- [19] G. Vidal, D. Jonathan, and M. A. Nielsen Phys. Rev. A **62**, 012304 (2000)
- [20] L. Vandenberghe and S. Boyd SIAM Review **38**, 49 (1996)
- [21] E. M. Rains IEEE Trans. Inf. Theory **47**, 2921 (2001), E. Rains Phys. Rev. A **60**, 173 (1999); *ibid* **60**, 179 (1999), J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein Phys. Rev. Lett. **86**, 544 (2001)
- [22] G. Chiribella and G. M. D'Ariano quant-ph/0603168, G. M. D'Ariano J. Math. Phys. **45**, 3620 (2004)
- [23] A. K. Pati Phys. Rev. A **66**, 062319 (2002)
- [24] <http://control.ee.ethz.ch/~joloef/yalmip.php>
- [25] <http://sedumi.mcmaster.ca/>
- [26] K. Kraus, States, Effects, and Operations: Fundamental Notions of Quantum Theory. Lecture Notes in Physics, **190** (Springer-Verlag, Berlin, 1983)
- [27] R. Demkowicz-Dobrzański, private communication
- [28] Decomposition (14) is not in a visible LOCC form but we know that Kraus maps are not unique [26] as well as the corresponding positive matrices in the Jamiolkowski representation. However, due to the PPT condition laid on $R_{\mathcal{M}}$ the particular PPT (for two-qubit states thus LOCC) Kraus decomposition can be derived.